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Differential realizations, boson–fermion realizations of the *spl*(2, 1) superalgebra and its representations

Yong-Qing Chen

Department of Physics, Hunan Normal University, Changsha 410006, People's Republic of China

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Abstract. Differential realizations of the *spl*(2, 1) superalgebra on the spaces of homogeneous and inhomogeneous polynomials and the corresponding boson–fermion realizations are studied. The new indecomposable and irreducible representations of the *spl*(2, 1) superalgebra are given on subspaces and quotient spaces of the universal enveloping algebra of Heisenberg–Weyl superalgebra. All the finite-dimensional irreducible representations of the *spl*(2, 1) superalgebra are naturally obtained as special cases.

1. Introduction

Lie superalgebras have become increasingly important in nuclear physics, superunification, and in supergravity [1–3]. Recently, Turbiner and Ushveridze [4] have discussed the quasi-exactly solvable problems in quantum mechanics. A connection of quasi-exactly solvable problems and finite-dimensional inhomogeneous differential realizations of Lie algebras (or superalgebras) has been described at the first time by Turbiner [5]. The key to the settlement of the quasi-exactly solvable problems lies in studying finite-dimensional inhomogeneous differential realizations of Lie superalgebras. The case of some superalgebras has been considered by Shifman and Turbiner [6] and recently by Turbiner [7]. This paper of Backhouse [8] has also described one way of obtaining differential realizations of superalgebras. In the present paper we shall be concerned with the *spl*(2, 1) superalgebra. The purpose of the present paper is to derive further inhomogeneous differential realization of the *spl*(2, 1) superalgebra on the space of inhomogeneous polynomials employing variable substitution technique on the basis of the homogeneous differential realization. We then consider their corresponding relations of C-number differential operators and boson creation and annihilation operators, of Grassmann number differential operators and fermion creation and annihilation operators respectively. The corresponding boson–fermion realizations of the *spl*(2, 1) superalgebra are obtained in terms of homogeneous and inhomogeneous differential realizations. The indecomposable representations of Lie superalgebras are well known to play a crucial role in describing unstable particle systems [9]. It is quite a valid approach to employ the boson–fermion realizations of Lie superalgebras in order to study their indecomposable representations [10–13]. In the present paper we shall study indecomposable representations of the *spl*(2, 1) superalgebra on the universal enveloping algebra of Heisenberg–Weyl superalgebra, and on its subspaces and quotient spaces using the inhomogeneous boson–fermion realization of this superalgebra. All the finite-dimensional irreducible representations of the $spl(2,1)$ superalgebra are naturally obtained as special cases on the subspaces of generalized Fock space.

2. Homogeneous differential realization and corresponding boson–fermion realization of the $spl(2,1)$

In accordance with Scheunert *et al* [14] the generators of the $spl(2,1)$ superalgebra read as follows:

$$\{Q_3, Q_+, Q_-, B \in spl(2,1)\bar{0} \,|\, V_+, V_-, W_+, W_- \in spl(2,1)\bar{1}\} \tag{2.1}$$

and satisfy the following commutation and anticommutation relations:

$$\begin{array}{ll} [Q_3, Q_\pm] = \pm Q_\pm & [Q_+, Q_-] = 2Q_3 \\ [B, Q_\pm] = [B, Q_3] = 0 & \\ [Q_3, V_\pm] = \pm\frac{1}{2}V_\pm & [Q_3, W_\pm] = \pm\frac{1}{2}W_\pm \\ [Q_\pm, V_\mp] = V_\pm & [Q_\pm, W_\mp] = W_\pm \\ [Q_\pm, V_\pm] = 0 & [Q_\pm, W_\pm] = 0 \\ [B, V_\pm] = \frac{1}{2}V_\pm & [B, W_\pm] = -\frac{1}{2}W_\pm \\ \{V_\pm, V_\pm\} = \{V_\pm, V_\mp\} = \{W_\pm, W_\pm\} = \{W_\pm, W_\mp\} = 0 & \\ \{V_\pm, W_\pm\} = \pm Q_\pm & \{V_\pm, W_\mp\} = -Q_3 \pm B. \end{array} \tag{2.2}$$

We choose a $(2,2)$ dimensional irreducible representation D:

$$D(Q_3) = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & -1/2 & & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad D(B) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$$D(Q_+) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad D(Q_-) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D(V_+) = \begin{bmatrix} 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 0 \end{bmatrix} \qquad D(V_-) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \\ -1/\sqrt{2} & 0 & 0 & 0 \end{bmatrix}$$

$$D(W_+) = \begin{bmatrix} 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad D(W_-) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.3}$$

In order to study differential realization of the *spl*(2,1) superalgebra on the space of homogeneous polynomials, introducing four independent variables μ_1, μ_2, ξ_1, ξ_2 where μ_1, μ_2 are C-numbers and ξ_1, ξ_2 are Grassmann numbers respectively, we regard them as the basis of representation space, i.e.

$$\mu_1=\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad \mu_2=\begin{bmatrix}0\\1\\0\\0\end{bmatrix} \qquad \xi_1=\begin{bmatrix}0\\0\\1\\0\end{bmatrix} \qquad \xi_2=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}. \tag{2.4}$$

Noting (2.3) and (2.4), we obtain

$$\begin{array}{llll}
Q_3\mu_1=\frac{1}{2}\mu_1 & Q_3\mu_2=-\frac{1}{2}\mu_2 & Q_3\xi_1=0 & Q_3\xi_2=0 \\
B_{\mu 1}=0 & B\mu_2=0 & B\xi_1=-\frac{1}{2}\xi_1 & B\xi_2=\frac{1}{2}\xi_2 \\
Q+\mu_1=0 & Q_+\mu_2=\mu_1 & Q_+\xi_1=0 & Q_+\xi_2=0 \\
Q_-\mu_1=\mu_2 & Q_-\mu_2=0 & Q_-\xi_1=0 & Q_-\xi_2=0 \\
V_+\mu_1=0 & V_+\mu_2=\frac{1}{\sqrt{2}}\xi_2 & V_+\xi_1=\frac{1}{\sqrt{2}}\mu_1 & V_+\xi_2=0 \\
V_-\mu_1=-\frac{1}{\sqrt{2}}\xi_2 & V_-\mu_2=0 & V_-\xi_1=\frac{1}{\sqrt{2}}\mu_2 & V_-\xi_2=0 \\
W_+\mu_1=0 & W_+\mu_2=\frac{1}{\sqrt{2}}\xi_1 & W_+\xi_1=0 & W_+\xi_2=\frac{1}{\sqrt{2}}\mu_1 \\
W_-\mu_1=\frac{-1}{\sqrt{2}}\xi_1 & W_-\mu_2=0 & W_-\xi_1=0 & W_-\xi_2=\frac{1}{\sqrt{2}}\mu_2.
\end{array} \tag{2.5}$$

Using differential operators the generators of the *spl*(2, 1) are constructed as follows:

$$\begin{array}{ll}
Q_3=\frac{1}{2}(\mu_1\partial/\partial\mu_1-\mu_2\partial/\partial\mu_2) & B=\frac{1}{2}(\xi_2\partial/\partial\xi_2-\xi_1\partial/\partial\xi_1) \\
Q_+=\mu_1\partial/\partial\mu_2 & Q_-=\mu_2\partial/\partial\mu_1 \\
V_+=\frac{1}{\sqrt{2}}(\mu_1\partial/\partial\xi_1+\xi_2\partial/\partial\mu_2) & V_-=-\frac{1}{\sqrt{2}}(\xi_2\partial/\partial\mu_1-\mu_2\partial/\partial\xi_1) \\
W_+=\frac{1}{\sqrt{2}}(\mu_1\partial/\partial\xi_2+\xi_1\partial/\partial\mu_2) & W_-=-\frac{1}{\sqrt{2}}(\xi_1\partial/\partial\mu_1-\mu_2\partial/\partial\xi_2).
\end{array} \tag{2.6}$$

It is easily proved that the generators thus represented satisfy all the commutation and anticommutation relations of the *spl*(2,1). Substantially, (2.6) is a differential realization on the space of homogeneous polynomials of degree one, i.e $A_1=\{\mu_1, \mu_2, \xi_1, \xi_2\}$. For the space of homogeneous polynomials of degree n,

$$A_n=\{\mu_1^{i_1}\mu_2^{i_2}\xi_1^{k_1}\xi_2^{k_2}|i_1, i_2\in Z^+, k_1, k_2=0.1 \text{ and } i_1+i_2+k_1+k_2=n\} \tag{2.7}$$

where Z^+ denotes the set of all non-negative integer, it carries the direct product representation of the $spl(2,1)$,

$$D_s^{\otimes_n} = (\underbrace{D\otimes D\otimes\cdots\otimes D}_{\text{degree } n}) \text{ symmetrized.} \tag{2.8}$$

Using the definition of direct product representation,

$$\begin{aligned}\hat{F}(\mu_1^{i_1}\mu_2^{i_2}\xi_1^{k_1}\xi_2^{k_2}) &= (F\mu_1^{i_1})\mu_2^{i_2}\xi_1^{k_1}\xi_2^{k_2} + \mu_1^{i_1}(F\mu_2^{i_2})\xi_1^{k_1}\xi_2^{k_2} \\ &\quad + \mu_1^{i_1}\mu_2^{i_2}(F\xi_1^{k_1})\xi_2^{k_2} + \mu_1^{i_1}\mu_2^{i_2}\xi_1^{k_1}(F\xi_2^{k_2})\end{aligned} \tag{2.9}$$

where F stands for any generator of the $spl(2.1)$, we can obtain its differential realization $\hat{F}$ on A_n. It is easy to check that $\hat{F} = F$.

Consider their corresponding relations of C-number differential operators $(\mu_i, \partial/\partial\mu_i)$ and boson creation and annihilation operators (b_i^+, b_i),

$$\begin{aligned}&b_i^+ \Leftrightarrow \mu_i \qquad && b_i \Leftrightarrow \partial/\partial\mu_i \\ &[b_i, b_j^+] = \delta_{ij} && [\partial/\partial\mu_i, \mu_j] = \delta_{ij} \\ &[b_i, b_j] = [b_i^+, b_j^+] = 0 && \\ &[\partial/\partial\mu_i, \partial/\partial\mu_j] = [\mu_i, \mu_j] = 0 &&\end{aligned} \tag{2.10}$$

and of Grassmann number differential operators $(\xi_i, \partial/\partial\xi_i)$ and fermion creation and annihilation operators (a_i^+, a_i), respectively,

$$\begin{aligned}&a_i^+ \Leftrightarrow \xi_i \qquad && a_i \Leftrightarrow \partial/\partial\xi_i \\ &\{a_i, a_j^+\} = \delta_{ij} && \{\partial/\partial\xi_i, \xi_j\} = \delta_{ij} \\ &\{a_i, a_j\} = \{a_i^+, a_j^+\} = 0 && \\ &\{\partial/\partial\xi_i, \partial/\partial\xi_j\} = \{\xi_i, \xi_j\} = 0. &&\end{aligned} \tag{2.11}$$

The corresponding homogeneous boson–fermion realization of the $spl(2,1)$ is obtained in terms of two pairs of boson operators and two pairs of fermion operators as follows:

$$\begin{aligned}&Q_3 = \tfrac{1}{2}(b_1^+b_1 - b_2^+b_2) \qquad && B = \tfrac{1}{2}(a_2^+a_2 - a_1^+a_1) \\ &Q_+ = b_1^+b_2 && Q_- = b_2^+b_1 \\ &V_+ = \frac{1}{\sqrt{2}}(b_1^+a_1 + a_2^+b_2) && V_- = -\frac{1}{\sqrt{2}}(a_2^+b_1 - b_2^+a_1) \\ &W_+ = \frac{1}{\sqrt{2}}(b_1^+a_2 + a_1^+b_2) && W_- = -\frac{1}{\sqrt{2}}(a_1^+b_1 - b_2^+a_2).\end{aligned} \tag{2.12}$$

3. Inhomogeneous differential realization and corresponding boson-fermion realization of the *spl*(2,1)

In order to get differential realization on the space of inhomogeneous polynomials, we introduce three new independent variables (x, y_1, y_2) and employ variable substitution

$$x = \frac{\mu_1}{\mu_2} \qquad y_1 = \frac{\xi_1}{\mu_2} \qquad y_2 = \frac{\xi_2}{\mu_2} \qquad (\mu_2 \neq 0) \tag{3.1}$$

where x is a C-number and y_1, y_2 are Grassmann numbers respectively. Clearly, the basis of A_n becomes

$$\mu_1^{i_1}\mu_2^{i_2}\xi_1^{k_1}\xi_2^{k_2} \Rightarrow x^{i_1}\mu_2^n y_1^{k_1} y_2^{k_2} \qquad i_1+k_1+k_2=0, 1, \ldots, n. \tag{3.2}$$

Let

$$A_n' = \{x^{i_1}\mu_2^n y_1^{k_1} y_2^{k_2} | i_1+k_1+k_2=0, 1, \ldots, n, i_1 \in Z^+, k_1, k_2=0, 1\} \tag{3.3}$$

then A_n' is a space of inhomogeneous polynomials.

Using (2.6), (3.1) and the following definition

$$\begin{aligned}\bar{F}(x^{i_1}\mu_2^n y_1^{k_1} y_2^{k_2}) &= (\hat{F}x^{i_1})\mu_2^n y_1^{k_1} y_2^{k_2} + x^{i_1}(\hat{F}\mu_2^n) y_1^{k_1} y_2^{k_2} + x^{i_1}\mu_2^n(\hat{F}y_1^{k_1}) y_2^{k_2} \\ &\quad + x^{i_1}\mu_2^n y_1^{k_1}(\hat{F}y_2^{k_2})\end{aligned} \tag{3.4}$$

we get the inhomogeneous differential realization $\bar{F}$ of the $spl(2,1)$ on A_n',

$$\begin{aligned}
\bar{Q}_3 &= -\tfrac{1}{2}n + x\partial/\partial x + \tfrac{1}{2}y_1\partial/\partial y_1 + \tfrac{1}{2}y_2\partial/\partial y_2 \\
\bar{B} &= \tfrac{1}{2}y_2\partial/\partial y_2 - \tfrac{1}{2}y_1\partial/\partial y_1 \\
\bar{Q}_+ &= nx - x^2\partial/\partial x - xy_1\partial/\partial y_1 - xy_2\partial/\partial y_2 \\
\bar{Q}_- &= \partial/\partial x \\
\bar{V}_+ &= \frac{1}{\sqrt{2}}ny_2 + \frac{1}{\sqrt{2}}x\partial/\partial y_1 - \frac{1}{\sqrt{2}}y_2x\partial/\partial x - \frac{1}{\sqrt{2}}y_2y_1\partial/\partial y_1 \\
\bar{V}_- &= -\frac{1}{\sqrt{2}}y_2\partial/\partial x + \frac{1}{\sqrt{2}}\partial/\partial y_1 \\
\bar{W}_+ &= \frac{1}{\sqrt{2}}ny_1 + \frac{1}{\sqrt{2}}x\partial/\partial y_2 - \frac{1}{\sqrt{2}}y_1x\partial/\partial x - \frac{1}{\sqrt{2}}y_1y_2\partial/\partial y_2 \\
\bar{W}_- &= -\frac{1}{\sqrt{2}}y_1\,\partial/\partial x + \frac{1}{\sqrt{2}}\partial/\partial y_2.
\end{aligned} \tag{3.5}$$

It is worthy of note that μ_2 is a cofactor in the basis of A_n'. Granted that we extend the non-negative integer n to any real number, one still gets (3.5).

With a similar way, considering their corresponding relations of C-number differential operators $(x, \partial/\partial x)$ and boson creation and annihilation operators (b^+, b), and of Grassmann number differential operators $(y_1, \partial/\partial y_1; y_2, \partial/\partial y_2)$ and fermion creation and annihilation operators $(a_1^+, a_1; a_2^+, a_2)$

$$b^+ \Leftrightarrow x \qquad b \Leftrightarrow \partial/\partial x \tag{3.6}$$

$$\begin{aligned} a_1^+ \Leftrightarrow y_1 &\qquad a_1 \Leftrightarrow \partial/\partial y_1 \\ a_2^+ \Leftrightarrow y_2 &\qquad a_2 \Leftrightarrow \partial/\partial y_2 \end{aligned} \tag{3.7}$$

we can get the corresponding inhomogeneous boson–fermion realization,

$$\begin{aligned}
\bar{Q}_3 &= -\tfrac{1}{2}n + b^+b + \tfrac{1}{2}a_1^+a_1 + \tfrac{1}{2}a_2^+a_2 \\
\bar{B} &= \tfrac{1}{2}a_2^+a_2 - \tfrac{1}{2}a_1^+a_1 \\
\bar{Q}_+ &= nb^+ - b^{+2}b - b^+a_1^+a_1 - b^+a_2^+a_2 \\
\bar{Q}_- &= b
\end{aligned} \tag{3.8}$$

$$\tilde{V}_+ = \frac{1}{\sqrt{2}} n a_2^+ + \frac{1}{\sqrt{2}} b^+ a_1 - \frac{1}{\sqrt{2}} a_2^+ b^+ b - \frac{1}{\sqrt{2}} a_2^+ a_1^+ a_1$$

$$\tilde{V}_- = -\frac{1}{\sqrt{2}} a_2^+ b + \frac{1}{\sqrt{2}} a_1$$

$$\tilde{W}_+ = \frac{1}{\sqrt{2}} n a_1^+ + \frac{1}{\sqrt{2}} b^+ a_2 - \frac{1}{\sqrt{2}} a_1^+ b^+ b - \frac{1}{\sqrt{2}} a_1^+ a_2^+ a_2$$

$$\tilde{W}_- = -\frac{1}{\sqrt{2}} a_1^+ b + \frac{1}{\sqrt{2}} a_2.$$

Obviously, we use only one pair of boson operators and two pairs of fermion operators in obtaining inhomogeneous boson–fermion realization.

4. Indecomposable representation of the $spl(2,1)$

Consider $(1+2)$ state Heisenberg–Weyl superalgebra H: $\{b^+, b, a_1^+, a_1, a_2^+, a_2, E\}$ where E stands for the unit operator. According to the Poincare–Birckhoff–Witt theorem, we choose for its universal enveloping algebra Ω a basis

$$\{\varphi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2, t) = b^{+k} b^l a_1^{+\alpha_1} a_1^{\beta_1} a_2^{+\alpha_2} a_2^{\beta_2} E^t | k, l, t \in Z^+, \alpha_1, \beta_1, \alpha_2, \beta_2 = 0, 1\}. \quad (4.1)$$

Each vector in the space of Ω is a linear combination of the basis with complex coefficients. Then, we consider an extension $\bar{\Omega}$ of the space Ω, in which each element is a linear combination of the basis whose coefficients are elements of the Grassmann algebra $\bar{G}$.

The representation of the superalgebra H on the space of $\bar{\Omega}$ is defined as

$$\begin{aligned}
f(b^+)\varphi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2, t) &= \varphi(k+1, l, \alpha_1, \beta_1, \alpha_2, \beta_2, t) \\
f(b)\varphi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2, t) & \\
&= \varphi(k, l+1, \alpha_1, \beta_1, \alpha_2, \beta_2, t) + k\varphi(k-1, l, \alpha_1, \beta_1, \alpha_2, \beta_2, t+1) \\
f(a_1^+)\varphi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2, t) &= (1-\alpha_1)\varphi(k, l, \alpha_1+1, \beta_1, \alpha_2, \beta_2, t) \qquad (4.2) \\
f(a_1)\varphi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2, t) & \\
&= (-1)^{\alpha_1}\varphi(k, l, \alpha_1, \beta_1+1, \alpha_2, \beta_2, t) + \alpha_1\varphi(k, l, \alpha_1-1, \beta_1, \alpha_2, \beta_2, t+1) \\
f(a_2^+)\varphi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2, t) &= (-1)^{\alpha_1+\beta_1}(1-\alpha_2)\varphi(k, l, \alpha_1, \beta_1, \alpha_2+1, \beta_2, t) \\
f(a_2)\varphi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2, t) &= (-1)^{\alpha_1+\beta_1+\alpha_2}\varphi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2+1, t) \\
&\quad + (-1)^{\alpha_1+\beta_1}\alpha_2\varphi(k, l, \alpha_1, \beta_1, \alpha_2-1, \beta_2, t+1).
\end{aligned}$$

Now, we consider the quotient space V with the basis

$$\begin{aligned}
V = (\Omega/I)&:\{\varphi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2) = \varphi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2, 0) \bmod I \\
&\times k, l \in Z^+, \alpha_1, \beta_1, \alpha_2, \beta_2 = 0, 1\} \qquad (4.3)
\end{aligned}$$

corresponding to the two-sided ideal I generated by the element $E-1$.

The representation (4.2) induces the new representation on the space of V

$$f(b^+)\varphi(k, l, \alpha_1, \beta_1, \alpha_2, \beta_2) = \varphi(k+1, l, \alpha_1, \beta_1, \alpha_2, \beta_2)$$

$$f(b)\varphi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)$$
$$=\varphi(k,l+1,\alpha_1,\beta_1,\alpha_2,\beta_2)+k\varphi(k-1,l,\alpha_1,\beta_1,\alpha_2,\beta_2)$$
$$f(a_1^+)\varphi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)=(1-\alpha_1)\varphi(k,l,\alpha_1+1,\beta_1,\alpha_2,\beta_2) \qquad (4.4)$$
$$f(a_1)\varphi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)$$
$$=(-1)^{\alpha_1}\varphi(k,l,\alpha_1,\beta_1+1,\alpha_2,\beta_2)+\alpha_1\varphi(k,l,\alpha_1-1,\beta_1,\alpha_2,\beta_2)$$
$$f(a_2^+)\varphi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)=(-1)^{\alpha_1+\beta_1}(1-\alpha_2)\varphi(k,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2)$$
$$f(a_2)\varphi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)=(-1)^{\alpha_1+\beta_1+\alpha_2}\varphi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2+1)$$
$$+(-1)^{\alpha_1+\beta_1}\alpha_2\varphi(k,l,\alpha_1,\beta_1,\alpha_2-1,\beta_2).$$

Using the following relation

$$L(F(b^+,b,a_1^+,a_1,a_2^+,a_2))=\bar{F}(f(b^+),f(b),f(a_1^+),f(a_1),f(a_2^+),f(a_2)) \qquad (4.5)$$

and the boson–fermion realization (3.8), we obtain the representation L of the $spl(2.1)$ on the space of V,

$$L(Q_3)\varphi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)=(-\tfrac{1}{2}n+k+\tfrac{1}{2}\alpha_1+\tfrac{1}{2}\alpha_2)\varphi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)$$
$$+\varphi(k+1,l+1,\alpha_1,\beta_1,\alpha_2,\beta_2)$$
$$+\tfrac{1}{2}(-1)^{\alpha_1}(1-\alpha_1)\varphi(k,l,\alpha_1+1,\beta_1+1,\alpha_2,\beta_2)$$
$$+\tfrac{1}{2}(-1)^{\alpha_2}(1-\alpha_2)\varphi(k,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2+1)$$
$$L(B)\varphi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)=(\tfrac{1}{2}\alpha_2-\tfrac{1}{2}\alpha_1)\,\varphi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)$$
$$+\tfrac{1}{2}(-1)^{\alpha_2}(1-\alpha_2)\varphi(k,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2+1)$$
$$-\tfrac{1}{2}(-1)^{\alpha_1}(1-\alpha_1)\varphi(k,l,\alpha_1+1,\beta_1+1,\alpha_2,\beta_2)$$
$$L(Q_+)\varphi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)=(n-k-\alpha_1-\alpha_2)\varphi(k+1,l,\alpha_1,\beta_1,\alpha_2,\beta_2)$$
$$-\varphi(k+2,l+1,\alpha_1,\beta_1,\alpha_2,\beta_2)$$
$$-(-1)^{\alpha_1}(1-\alpha_1)\varphi(k+1,l,\alpha_1+1,\beta_1+1,\alpha_2,\beta_2)$$
$$-(-1)^{\alpha_2}(1-\alpha_2)\varphi(k+1,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2+1) \qquad (4.6)$$
$$L(Q_-)\varphi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)$$
$$=\varphi(k,l+1,\alpha_1,\beta_1,\alpha_2,\beta_2)+k\varphi(k-1,l,\alpha_1,\beta_1,\alpha_2,\beta_2)$$
$$L(V_+)\varphi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)$$
$$=\frac{1}{\sqrt{2}}(-1)^{\alpha_1+\beta_1}(n-k-\alpha_1)(1-\alpha_2)\varphi(k,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2)$$
$$+\frac{1}{\sqrt{2}}(-1)^{\alpha_1}\varphi(k+1,l,\alpha_1,\beta_1+1,\alpha_2,\beta_2)$$
$$+1\frac{1}{\sqrt{2}}\alpha_1\varphi(k+1,l,\alpha_1-1,\beta_1,\alpha_2,\beta_2)$$
$$-\frac{1}{\sqrt{2}}(-1)^{\alpha_1+\beta_1}(1-\alpha_2)\varphi(k+1,l+1,\alpha_1,\beta_1,\alpha_2+1,\beta_2)$$

$$-\frac{1}{\sqrt{2}}(-1)^{\beta_1}(1-\alpha_1)(1-\alpha_2)\varphi(k+1,l+1,\alpha_1,\beta_1,\alpha_2+1,\beta_2)$$

$$L(V_-)\varphi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)=\frac{1}{\sqrt{2}}(-1)^{\alpha_1}\varphi(k,l,\alpha_1,\beta_1+1,\alpha_2,\beta_2)$$

$$+\frac{1}{\sqrt{2}}\alpha_1\varphi(k,l,\alpha_1-1,\beta_1,\alpha_2,\beta_2)$$

$$-\frac{1}{\sqrt{2}}(-1)^{\alpha_1+\beta_1}(1-\alpha_2)\varphi(k,l+1,\alpha_1,\beta_1,\alpha_2+1,\beta_2)$$

$$-\frac{1}{\sqrt{2}}(-1)^{\alpha_1+\beta_1}k(1-\alpha_2)\varphi(k-1,l,\alpha_1,\beta_1,\alpha_2+1,\beta_2)$$

$$L(W_+)\varphi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)=\frac{1}{\sqrt{2}}(n-k-a_2)(1-\alpha_1)\varphi(k,l,\alpha_1+1,\beta_1,\alpha_2,\beta_2)$$

$$+\frac{1}{\sqrt{2}}(-1)^{\alpha_1+\beta_1+\alpha_2}\varphi(k+1,l,\alpha_1,\beta_1,\alpha_2,\beta_2+1)$$

$$+\frac{1}{\sqrt{2}}(-1)^{\alpha_1+\beta_1}\alpha_2\varphi(k+1,l,\alpha_1,\beta_1,\alpha_2-1,\beta_2)$$

$$-\frac{1}{\sqrt{2}}(1-\alpha_1)\varphi(k+1,l+1,\alpha_1+1,\beta_1,\alpha_2,\beta_2)$$

$$-\frac{1}{\sqrt{2}}(-1)^{\alpha_2}(1-\alpha_1)(1-\alpha_2)\varphi(k,l,\alpha_1+1,\beta_1,\alpha_2+1,\beta_2+1)$$

$$L(W_-)\varphi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)=\frac{1}{\sqrt{2}}(-1)^{\alpha_1+\beta_1+\alpha_2}\varphi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2+1)$$

$$+\frac{1}{\sqrt{2}}(-1)^{\alpha_1+\beta_1}\alpha_2\varphi(k,l,\alpha_1,\beta_1,\alpha_2-1,\beta_2)$$

$$-\frac{1}{\sqrt{2}}(1-\alpha_1)\varphi(k,l+1,\alpha_1+1,\beta_1,\alpha_2,\beta_2)$$

$$-\frac{1}{\sqrt{2}}k(1-\alpha_1)\varphi(k-1,l,\alpha_1+1,\beta_1,\alpha_2,\beta_2).$$

From (4.6), it follows that the sum $(l+\beta_1+\beta_2)$ does not decrease under the action of the representation L and the subspace

$$V_m=\{\varphi(k,l,\alpha_1,\beta_1,\alpha_2,\beta_2)\in V|l+\beta_1+\beta_2\geq m\}$$

is invariant, for which no invariant complementary subspace exists. Thus, the representation given by (4.6) on the space V is indecomposable.

The generalized Fock space is defined as a quotient space of V

$$Y=(V/J):\{\varphi(k,\alpha_1,\alpha_2)=\varphi(k,0,\alpha_1,0,\alpha_2,0)\bmod J|k\in Z^+,\alpha_1,\alpha_2=0,1\}$$

where J is the left ideal generated by the element $b-\lambda$, $\alpha_1-\eta_1$ and $a_1-\eta_2$, λ is a complex number and η_1 and η_2 are generators of the Grassmann algebra $\tilde{G}$. On this space, the representation (4.6) induces the new representation

$$\begin{aligned}
L(Q_s)\varphi(k,\alpha_1,\alpha_2) &= \left(-\frac{1}{2}n+k+\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2\right)\varphi(k,\alpha_1,\alpha_2) \\
&\quad +\lambda\varphi(k+1,\alpha_1,\alpha_2)+\frac{1}{2}(1-\alpha_1)\eta_1\varphi(k,\alpha_1+1,\alpha_2) \\
&\quad +\frac{1}{2}(1-\alpha_2)\eta_2\varphi(k,\alpha_1,\alpha_2+1) \\
L(B)\varphi(k,\alpha_1,\alpha_2) &= \left(\frac{1}{2}\alpha_2-\frac{1}{2}\alpha_1\right)\varphi(k,\alpha_1,\alpha_2)+\frac{1}{2}(1-\alpha_2)\eta_2\varphi(k,\alpha_1,\alpha_2+1) \\
&\quad -\frac{1}{2}(1-\alpha_1)\eta_1\varphi(k,\alpha_1+1,\alpha_2) \\
L(Q_+)\varphi(k,\alpha_1,\alpha_2) &= (n-k-\alpha_1-\alpha_2)\varphi(k+1,\alpha_1,\alpha_2)-\lambda\varphi(k+2,\alpha_1,\alpha_2) \\
&\quad -(1-\alpha_1)\eta_1\varphi(k+1,\alpha_1+1,\alpha_2)-(1-\alpha_2)\eta_2\varphi(k+1,\alpha_1,\alpha_2+1) \\
L(Q_-)\varphi(k,\alpha_1,\alpha_2) &= \lambda\varphi(k,\alpha_1,\alpha_2)+k\varphi(k-1,\alpha_1,\alpha_2) \qquad (4.7) \\
L(V_+)\varphi(k,\alpha_1,\alpha_2) &= \frac{1}{\sqrt{2}}(-1)^{\alpha_1}(n-k-\alpha_1)(1-\alpha_2)\varphi(k,\alpha_1,\alpha_2+1) \\
&\quad +\frac{1}{\sqrt{2}}(-1)^{\alpha_1}\eta_1\varphi(k+1,\alpha_1,\alpha_2)+\frac{1}{\sqrt{2}}\alpha_1\varphi(k+1,\alpha_1-1,\alpha_2) \\
&\quad -\frac{1}{\sqrt{2}}(-1)^{\alpha_1}(1-\alpha_2)\lambda\varphi(k+1,\alpha_1,\alpha_2+1) \\
&\quad -\frac{1}{\sqrt{2}}(1-\alpha_1)(1-\alpha_2)\eta_1\varphi(k,\alpha_1+1,\alpha_2+1) \\
L(V_-)\varphi(k,\alpha_1,\alpha_2) &= \frac{1}{\sqrt{2}}(-1)^{\alpha_1}\eta_1\varphi(k,\alpha_1,\alpha_2)+\frac{1}{\sqrt{2}}\alpha_1\varphi(k,\alpha_1-1,\alpha_2) \\
&\quad -\frac{1}{\sqrt{2}}(-1)^{\alpha_1}(1-\alpha_2)\lambda\varphi(k,\alpha_1,\alpha_2+1)-\frac{1}{\sqrt{2}}(-1)^{\alpha_1}(1-\alpha_2)k\varphi(k-1,\alpha_1,\alpha_2+1) \\
L(W_+)\varphi(k,\alpha_1,\alpha_2) &= \frac{1}{\sqrt{2}}(n-k-\alpha_2)(1-\alpha_1)\varphi(k,\alpha_1+1,\alpha_2) \\
&\quad +\frac{1}{\sqrt{2}}(-1)^{\alpha_1+\alpha_2}\eta_2\varphi(k+1,\alpha_1,\alpha_2)+\frac{1}{\sqrt{2}}(-1)^{\alpha_1}\alpha_2\varphi(k+1,\alpha_1,\alpha_2-1) \\
&\quad -\frac{1}{\sqrt{2}}(1-\alpha_1)\lambda\varphi(k+1,\alpha_1+1,\alpha_2) \\
&\quad -\frac{1}{\sqrt{2}}(-1)^{\alpha_2}(1-\alpha_1)(1-\alpha_2)\eta_2\varphi(k,\alpha_1+1,\alpha_2+1) \\
L(W_-)\varphi(k,\alpha_1,\alpha_2) &= \frac{1}{\sqrt{2}}(-1)^{\alpha_1+\alpha_2}\eta_2\varphi(k,\alpha_1,\alpha_2)+\frac{1}{\sqrt{2}}(-1)^{\alpha_1}\alpha_2\varphi(k,\alpha_1,\alpha_2-1)
\end{aligned}$$

$$-\frac{1}{\sqrt{2}}(1-\alpha_1)\lambda\varphi(k,\alpha_1+1,\alpha_2)-\frac{1}{\sqrt{2}}(1-\alpha_1)k\varphi(k-1,\alpha_1+1,\alpha_2).$$

The representation given by (4.7) is an infinite-dimensional irreducible representation for the cases $\lambda\neq 0$, $\eta_1\neq 0$ or $\eta_2\neq 0$. When $\lambda=0=\eta_1=\eta_2$, the representation (4.7) becomes

$$L(Q_3)\varphi(k,\alpha_1,\alpha_2)=\left(-\frac{1}{2}n+k+\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2\right)\varphi(k,\alpha_1,\alpha_2)$$

$$L(B)\varphi(k,\alpha_1,\alpha_2)=\left(\frac{1}{2}\alpha_2-\frac{1}{2}\alpha_1\right)\varphi(k,\alpha_1,\alpha_2)$$

$$L(Q_+)\varphi(k,\alpha_1,\alpha_2)=(n-k-\alpha_1-\alpha_2)\varphi(k+1,\alpha_1,\alpha_2)$$

$$L(Q_-)\varphi(k,\alpha_1,\alpha_2)=k\varphi(k-1,\alpha_1,\alpha_2) \tag{4.8}$$

$$L(V_+)\varphi(k,\alpha_1,\alpha_2)$$

$$=\frac{1}{\sqrt{2}}(-1)^{\alpha_1}(n-k-\alpha_1)(1-\alpha_2)\varphi(k,\alpha_1,\alpha_2+1)+\frac{1}{\sqrt{2}}\alpha_1\varphi(k+1,\alpha_1-1,\alpha_2)$$

$$L(V_-)\varphi(k,\alpha_1,\alpha_2)=\frac{1}{\sqrt{2}}\alpha_1\varphi(k,\alpha_1-1,\alpha_2)-\frac{1}{\sqrt{2}}(-1)^{\alpha_1}(1-\alpha_2)k\varphi(k-1,\alpha_1,\alpha_2+1)$$

$$L(W_+)\varphi(k,\alpha_1,\alpha_2)$$

$$=\frac{1}{\sqrt{2}}(n-k-\alpha_2)(1-\alpha_1)\varphi(k,\alpha_1+1,\alpha_2)+\frac{1}{\sqrt{2}}(-1)^{\alpha_1}\alpha_2\varphi(k+1,\alpha_1,\alpha_2-1)$$

$$L(W_-)\varphi(k,\alpha_1,\alpha_2)=\frac{1}{\sqrt{2}}(-1)^{\alpha_1}\alpha_2\varphi(k,\alpha_1,\alpha_2-1)-\frac{1}{\sqrt{2}}(1-\alpha_1)k\varphi(k-1,\alpha_1+1,\alpha_2).$$

We can easily see that the representation (4.8) is an infinite-dimensional irreducible representation when $n\notin Z^+$. Obviously, the invariant subspace exists when $n\in Z^+$,

$$Y(n)\colon \{\varphi(k,\alpha_1,\alpha_2)\in Y | k+\alpha_1+\alpha_2\leq n,\ k\in Z^+,\ \alpha_1,\alpha_2=0,1\}$$

$$\dim Y(n)=4n \tag{4.9}$$

and there is no invariant complementary subspace. Thus, the representation (4.8) is indecomposable. Restricting the representation given by (4.8) to the invariant subspace $Y(n)$, we can obtain a finite-dimensional irreducible representation of the *spl*(2,1). We shall discuss it in detail in the next section.

5. Finite-dimensional irreducible representation of the *spl*(2,1)

For the sake of simplicity, we re-define the basis of $Y(n)$ as

$$|j,m,\alpha_1,\alpha_2\rangle=1|[(j+m)!(j-m-\alpha_1)!(j-m-\alpha_2)!]^{1/2}\varphi(j+m,\alpha_1,\alpha_2) \tag{5.1}$$

$$j=(\tfrac{1}{2})\,n=0,\tfrac{1}{2},1,\ldots$$

$$m=-j,-j+1,\ldots,j \qquad \text{when } \alpha_1=0,\ \alpha_2=0$$

$m=-j,-j+1,\ldots,j-1$ when $\alpha_1=0,\alpha_2=1$

$m=-j,-j+1,\ldots,j-1$ when $\alpha_1=1,\alpha_2=0$

$m=-j,-j+1,\ldots,j-2$ when $\alpha_1=1,\alpha_2=1.$

The action of the generators of the *spl*(2,1) on the new basis vector is straightforwardly obtained with the help of (5.1) and (4.8). One finds

$$L(Q_3)|j,m,\alpha_1,\alpha_2\rangle=\left(m+\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2\right)|j,m,\alpha_1,\alpha_2\rangle$$

$$L(B)|j,m,\alpha_1,\alpha_2\rangle=\frac{1}{2}(\alpha_2-\alpha_2)|j,m,\alpha_1,\alpha_2\rangle$$

$$\begin{aligned}L(Q_+)|j,m,\alpha_1,\alpha_2\rangle&=(j-m-\alpha_1-\alpha_2)\{(j+m+1)\\&\quad/[(j-m-\alpha_1)(j-m-\alpha_2)]\}^{1/2}|j,m+1,\alpha_1,\alpha_2\rangle\end{aligned}\tag{5.2}$$

$$L(Q_-)|j,m,\alpha_1,\alpha_2\rangle=[(j+m)(j-m+1-\alpha_1)(j-m+1-\alpha_2)]^{1/2}|j,m-1,\alpha_1,\alpha_2\rangle$$

$$\begin{aligned}L(V_+)|j,m,\alpha_1,\alpha_2\rangle&\\=\frac{1}{\sqrt{2}}&(-1)^{\alpha_1}(1-\alpha_2)(j-m-\alpha_1)(j-m-\alpha_2)^{-1/2}|j,m,\alpha_1,\alpha_2+1\rangle\\&+\frac{1}{\sqrt{2}}\alpha_1[(j+m+1)/(j-m-\alpha_2)]^{1/2}|j,m+1,\alpha_1-1,\alpha_2\rangle\end{aligned}$$

$$\begin{aligned}L(V_-)|j,m,\alpha_1,\alpha_2\rangle&=\frac{1}{\sqrt{2}}\alpha_1(j-m+1-\alpha_1)^{1/2}|j,m,\alpha_1-1,\alpha_2\rangle\\&\quad-\frac{1}{\sqrt{2}}(-1)^{\alpha_1}(1-\alpha_2)[(j-m+1-\alpha_1)(j+m)]^{1/2}|j,m-1,\alpha_1,\alpha_2+1\rangle\end{aligned}$$

$$\begin{aligned}L(W_+)|j,m,\alpha_1,\alpha_2\rangle&=\frac{1}{\sqrt{2}}(1-\alpha_1)(j-m-\alpha_2)(j-m-\alpha_1)^{-1/2}j,m,\alpha_1+1,\alpha_2\rangle\\&\quad+\frac{1}{\sqrt{2}}(-1)^{\alpha_1}\alpha_2[(j+m+1)/(j-m-\alpha_1)]^{1/2}|j,m+1,\alpha_1,\alpha_2-1\rangle\end{aligned}$$

$$\begin{aligned}L(W_-)|j,m,\alpha_1,\alpha_2\rangle&=\frac{1}{\sqrt{2}}(-1)^{\alpha_1}\alpha_2(j-m+1-\alpha_2)^{1/2}|j,m,\alpha_1,\alpha_2-1\rangle\\&\quad-\frac{1}{\sqrt{2}}(1-\alpha_1)[(j-m+1-\alpha_2)(j+m)]^{1/2}|j,m-1,\alpha_1+1,\alpha_2\rangle.\end{aligned}$$

It is fairly straightforward to check that the above irreducible representation of the *spl*(2,1) has a finite dimension $8j$.

As an example, we write $(2+2)$ dimension irreducible representation of the *spl*(2,1), when $j=\frac{1}{2}$, that is,

$$L(Q_3)=\begin{bmatrix}1/2&0&0&0\\0&-1/2&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\qquad L(B)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&-1/2&0\\0&0&0&1/2\end{bmatrix}$$

$$L(Q_+)=\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$L(Q_-)=\begin{bmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\qquad L(V_+)=\begin{bmatrix}0&0&1/\sqrt{2}&0\\0&0&0&0\\0&0&0&0\\0&1/\sqrt{2}&0&0\end{bmatrix}$$

$$L(V_-)=\begin{bmatrix}0&0&0&0\\0&0&1/\sqrt{2}&0\\0&0&0&0\\-1/\sqrt{2}&0&0&0\end{bmatrix}$$

$$L(W_+)=\begin{bmatrix}0&0&0&1/\sqrt{2}\\0&0&0&0\\0&1/\sqrt{2}&0&0\\0&0&0&0\end{bmatrix}\qquad L(W_-)=\begin{bmatrix}0&0&0&0\\0&0&0&1/\sqrt{2}\\-1/\sqrt{2}&0&0&0\\0&0&0&0\end{bmatrix}. \tag{5.3}$$

It is easy to see that this $(2+2)$ dimension irreducible representation L is substantially the representation D that is chosen for constructing inhomogeneous differential realization of the $spl(2,1)$.

We have obtained the homogeneous and inhomogeneous differential realizations, the corresponding boson–fermion realizations of the $spl(2,1)$ and its indecomposable and irreducible represenations. All the finite-dimensional irreducible representations of the $spl(2,1)$ can be naturally obtained on the subspace of the generalized Fock space. Our method can be generalized to any Lie algebra (or superalgebra).

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